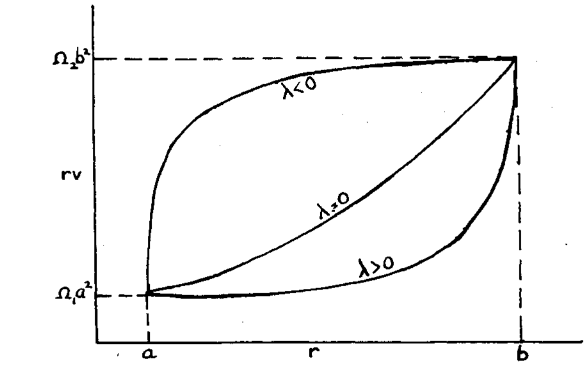

Fig. 1 The circulation for various λ, showing the presence of a boundary layer for large |λ|

having been discussed by Preston.[1] The purpose of this note is simply to bring this class of solutions to the attention of those engaged in the teaching of fluid dynamics.

The fluid is incompressible and is bounded by two long cylinders of radii a and b, rotating with angular velocities Ω_1 and Ω_2. The walls of the cylinders are porous, and fluid is emitted by one cylinder and absorbed by the other at equal rates. The motion depends on only one space coordinate, the radial distance r, and there are two velocity components, u radial and v transverse. The reason for the existence of a simple, exact solution is that the radial velocity is unaffected by the transverse velocity; but the reverse is not true, which provides the required interaction between viscous and inertial forces. The equations of motion and continuity are

$$u\frac{du}{dr} - \frac{v^2}{r} = -\frac{dp}{\rho dr} + \nu\left(\frac{d^2u}{dr^2} + \frac{du}{rdr} - \frac{u}{r^2}\right) \tag{1}$$

$$u\frac{dv}{dr} + \frac{uv}{r} = \nu\left(\frac{d^2v}{dr^2} + \frac{dv}{rdr} - \frac{v}{r^2}\right) \tag{2}$$

$$(d/dr)(ru) = 0 \tag{3}$$

These equations have a solution

$$u = m/r \tag{4}$$

$$V = (A/r) + Br^{1+\lambda} \tag{5}$$

where m is the strength of the source in the inner cylinder (negative if the inner cylinder is absorbing fluid), and $\lambda = m/\nu$; $|\lambda|$ is the Reynolds number of the radial motion. If $\lambda = -2$, the expression for v becomes

$$v = (A/r) + (B/r)\log r \tag{6}$$

The boundary conditions are $v = \Omega_1 a$ on $r = a$, and $v = \Omega_2 b$ on $r = b$; these determine the coefficients A and B to be

$$A = \frac{a^2b^2(\Omega_1 b^\lambda - \Omega_2 a^\lambda)}{b^{2+\lambda} - a^{2+\lambda}} \qquad B = \frac{\Omega_2 b^2 - \Omega_1 a^2}{b^{2+\lambda} - a^{2+\lambda}} \tag{7}$$

and for the exceptional case $\lambda = -2$,

$$A = \frac{\Omega_1 a^2 \log b - \Omega_2 b^2 \log a}{\log b - \log a} \qquad B = \frac{\Omega_2 b^2 - \Omega_1 a^2}{\log b - \log a} \tag{8}$$

The circulation rv has a typical behavior as indicated in Fig. 1, from which the formation of a boundary layer as the Reynolds number increases may be seen. For large Reynolds numbers, the circulation remains constant as it is convected across the region between the cylinders until it reaches the cylinder that is absorbing fluid, where viscous forces produce a sudden change in the value determined by the boundary condition. The division of the flow into a potential flow region and a boundary layer thus is demonstrated clearly.

Other solutions of a similar type which the reader (or his students) will easily obtain are provided by introducing a pressure gradient around the annulus with the cylinders at rest (flow in a curved channel), or by making the cylinders move at different rates parallel to their axes, or by introducing a pressure gradient in this direction. These four types can all be superimposed. Limiting cases when the outer cylinder tends to infinity, or the inner one shrinks to zero, provide some interesting results.[1] Another set of limiting cases is provided by keeping the difference in the radii fixed and letting the radii tend to infinity so that flow between parallel walls is obtained. All these solutions provide useful material for classroom discussion, but their common feature is the change in velocity profile to a boundary layer type as the Reynolds number increases, which the author feels is important to be able to demonstrate in a simple manner.

[1] Preston, J. H., "The steady circulatory flow about a circular cylinder with uniformly distributed suction at the surface," Aeronaut. Quart. 1, 319–338 (1950).

Long Circular-Cylindrical Shells Subjected to Circumferential, Radial Line Loads

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In Ref. 1, expressions for stresses and displacements of a long circular-cylindrical shell subjected to a uniform, external, circumferential, radial line load are derived by using complex Fourier transform of Love's stress function ϕ for an axisymmetrical problem, a technique used by Tranter and Craggs.[2] In this note the solution is derived by a direct procedure similar to the solution given for a solid circular cylinder in the recent handbook edited by Flügge.[3]

Governing Equations

It is required to find a stress function ϕ to satisfy the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \phi = 0 \tag{1}$$

for the cylinder shown in Fig. 1, with the additional conditions along the internal boundary C_i

$$\int_{C_i} du = 0 \qquad \int_{C_i} dv = 0 \qquad \int_{C_i} dw = 0 \tag{2}$$

The stresses and displacements are given by

$$\begin{aligned}
\sigma_r &= (\partial/\partial z)[\nu\nabla^2 - (\partial^2/\partial r^2)]\phi \\
\tau_{rz} &= (\partial/\partial r)[(1-\nu)\nabla^2 - (\partial^2/\partial z^2)]\phi \\
\sigma_\theta &= (\partial/\partial z)[\nu\nabla^2 - (1/r)(\partial/\partial r)]\phi \\
\sigma_z &= (\partial/\partial z)[(2-\nu)\nabla^2 - (\partial^2/\partial z^2)]\phi \\
u &= [(1+\nu)/E][2(1-\nu)\nabla^2 - (\partial^2/\partial z^2)]\phi \\
w &= [-(1+\nu)/E](\partial^2\phi/\partial r\partial z)
\end{aligned} \tag{3}$$

The boundary conditions are, when

$$r = b \qquad \tau_{rz} = 0 \text{ and } \sigma_r = 0 \tag{4}$$

and when

$$r = a \qquad \tau_{rz} = 0 \text{ and } \sigma_r = -f(z)$$

where $f(z)$ is the applied loading on the outer boundary, as-

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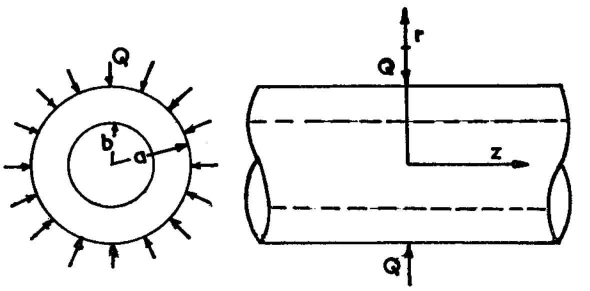

Fig. 1 Long circular-cylindrical shell

suming it as compressive. When the external load is of intensity p over a length 2ξ (in the z direction), $f(z)$ can be expressed as a Fourier-integral, i.e.,

$$f(z) = \frac{2p}{\pi} \int_0^\infty \frac{\sin\alpha\xi}{\alpha} \cos\alpha z d\alpha \tag{5}$$

For a concentrated radial line load Q, one has

$$f(z) = \frac{Q}{\pi} \int_0^\infty \cos\alpha z d\alpha \tag{6}$$

Following the approach for a solid cylinder,[3] the expression for ϕ can be written as follows:

$$\phi = \int_0^\infty \frac{1}{\alpha^3} [A(\alpha)I_0(\alpha r) + B(\alpha)\alpha r I_1(\alpha r) + C(\alpha)K_0(\alpha r) + D(\alpha)\alpha r K_1(\alpha r)] \sin\alpha z d\alpha \tag{7}$$

The stresses are

$$\sigma_r = \int_0^\infty \left\{ [(2\nu - 1)B(\alpha) - A(\alpha)]I_0(\alpha r) + \left[\frac{A(\alpha)}{\alpha r} - B(\alpha)\alpha r\right] I_1(\alpha r) + [(1 - 2\nu)D(\alpha) - C(\alpha)]K_0(\alpha r) - \left[\frac{C(\alpha)}{\alpha r} + D(\alpha)\alpha r\right] K_1(\alpha r) \right\} \cos\alpha z d\alpha$$

$$\tau_{rz} = \int_0^\infty \{ [A(\alpha) + 2(1 - \nu)B(\alpha)]I_1(\alpha r) + B(\alpha)\alpha r I_0(\alpha r) - [C(\alpha) - 2(1 - \nu)D(\alpha)]K_1(\alpha r) - D(\alpha)\alpha r K_0(\alpha r)\} \sin\alpha z d\alpha \tag{8}$$

$$\sigma_z = \int_0^\infty \{ [A(\alpha) + 2(2 - \nu)B(\alpha)]I_0(\alpha r) + B(\alpha)\alpha r I_1(\alpha r) + [C(\alpha) - 2(2 - \nu)D(\alpha)]K_0(\alpha r) + D(\alpha)\alpha r K_1(\alpha r)\} \cos\alpha z\, d\alpha$$

$$\sigma_\theta = \int_0^\infty \left[-A(\alpha) \frac{I_1(\alpha r)}{\alpha r} - (1 - \nu)B(\alpha)I_0(\alpha r) + C(\alpha)\frac{K_1(\alpha r)}{\alpha r} + (1 - \nu)D(\alpha)K_0(\alpha r) \right] \cos\alpha z d\alpha$$

and the displacements are

$$u = \frac{(1 + \nu)}{E} \int_0^\infty \frac{1}{\alpha} \{A(\alpha)I_0(\alpha r) + B(\alpha)[4(1 - \nu)I_0(\alpha r) + \alpha r I_1(\alpha r)] + C(\alpha)K_0(\alpha r) + D(\alpha)[\alpha r K_1(\alpha r) - 4(1 - \nu)K_0(\alpha r)]\} \sin\alpha z d\alpha$$

$$w = \frac{-(1 + \nu)}{E} \int_0^\infty \frac{1}{\alpha} [A(\alpha)I_1(\alpha r) + B(\alpha)\alpha r I_0(\alpha r) - C(\alpha)K_1(\alpha r) - D(\alpha)\alpha r K_0(\alpha r)] \cos\alpha z d\alpha \tag{9}$$

Substitution of the stresses from Eq. (3) into the boundary conditions given in Eqs. (4) and (6) gives the following simultaneous equations, put in the matrix form, to determine $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $D(\alpha)$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \\ D(\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_{45} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned}
a_{11} &= I_1(\alpha a) \\
a_{21} &= I_1(\alpha b) \\
a_{31} &= \frac{I_1(\alpha b)}{\alpha b} - I_0(\alpha b) \\
a_{41} &= \frac{I_1(\alpha a)}{\alpha a} - I_0(\alpha a) \\
a_{13} &= -K_1(\alpha a) \\
a_{23} &= -K_1(\alpha b) \\
a_{33} &= -K_0(\alpha b) - \frac{K_1(\alpha b)}{\alpha b} \\
a_{43} &= -K_0(\alpha a) - \frac{K_1(\alpha a)}{\alpha a} \\
a_{12} &= 2(1 - \nu)I_1(\alpha a) + \alpha a I_0(\alpha a) \\
a_{22} &= 2(1 - \nu)I_1(\alpha b) + \alpha b I_0(\alpha b) \\
a_{32} &= (2\nu - 1)I_0(\alpha b) - \alpha b I_1(\alpha b) \\
a_{42} &= (2\nu - 1)I_0(\alpha a) - \alpha a I_1(\alpha a) \\
a_{14} &= 2(1 - \nu)K_1(\alpha a) - \alpha a K_0(\alpha a) \\
a_{24} &= 2(1 - \nu)K_1(\alpha b) - \alpha b K_0(\alpha b) \\
a_{34} &= (1 - 2\nu)K_0(\alpha b) - \alpha b K_1(\alpha b) \\
a_{44} &= (1 - 2\nu)K_0(\alpha a) - \alpha a K_1(\alpha a) \\
a_{45} &= -Q/\pi
\end{aligned} \tag{11}$$

It can be seen easily that the expressions for stresses and displacements using Eqs. (3) and (10) will be the same as given by Eq. (21) in Ref. 1. Numerical integration to evaluate these stresses and displacements can be done as explained in Ref. 1.

References

[1] Klosner, J. M., "The elasticity solution of a long circular-cylindrical shell subjected to a uniform, circumferential, radial line load," J. Aerospace Sci. **29**, 834–841 (1962).

[2] Tranter, C. J. and Craggs, J. W., "The stress distribution in a long circular cylinder when a discontinuous pressure is applied to the curved surface," Phil. Mag. **36**, 241–250 (1945).

[3] Flügge, W. (ed.), *Handbook of Engineering Mechanics* (McGraw-Hill Book Co., Inc., New York, 1962), pp. 41-15–41-16.

Axisymmetric, Transverse Vibrations of a Spinning Membrane Clamped at Its Center

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By a simple change of scale depending only on Poisson's ratio and membrane geometry, it is shown that the axisymmetric transverse modes of vibration of a fully clamped membrane are equivalent to those of a partially clamped membrane.

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